Estimation Efficiency under A Semiparametric Density Ratio Model

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Outline

- Motivation
- A Semiparametric Model: Density Ratio Model
- Estimation Efficiency under the Density Ratio Model
  - Quantile Estimation

Motivation

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- In many disciplines, data are collected as multiple samples from **similar and connected** populations:

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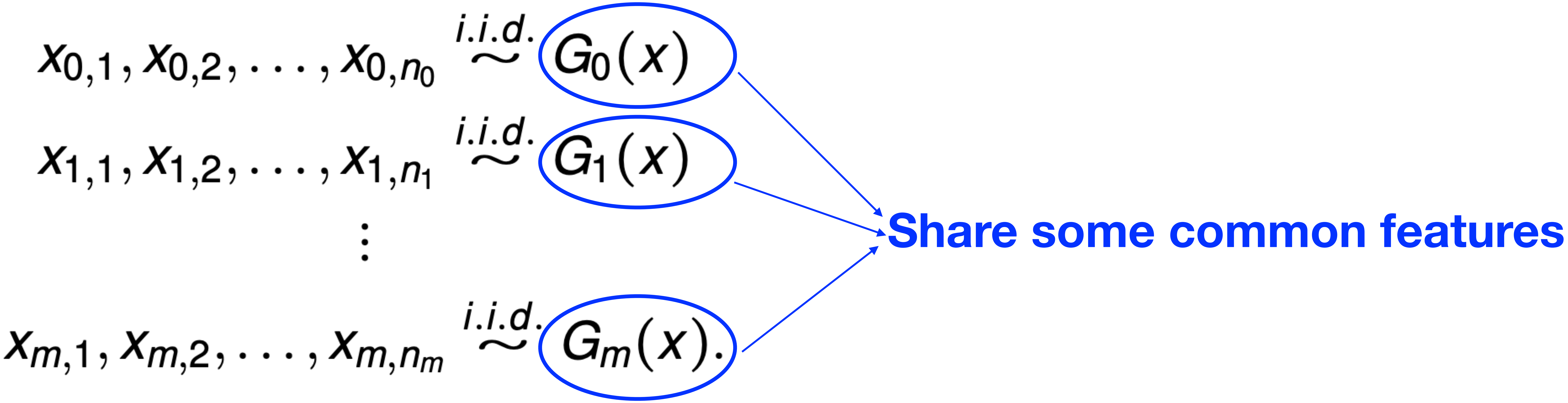

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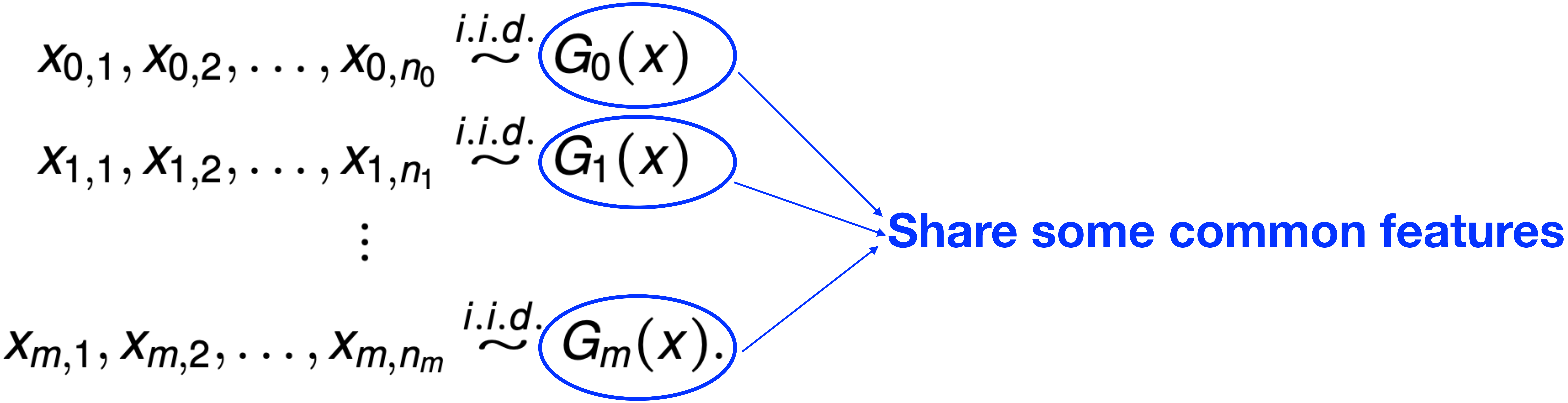

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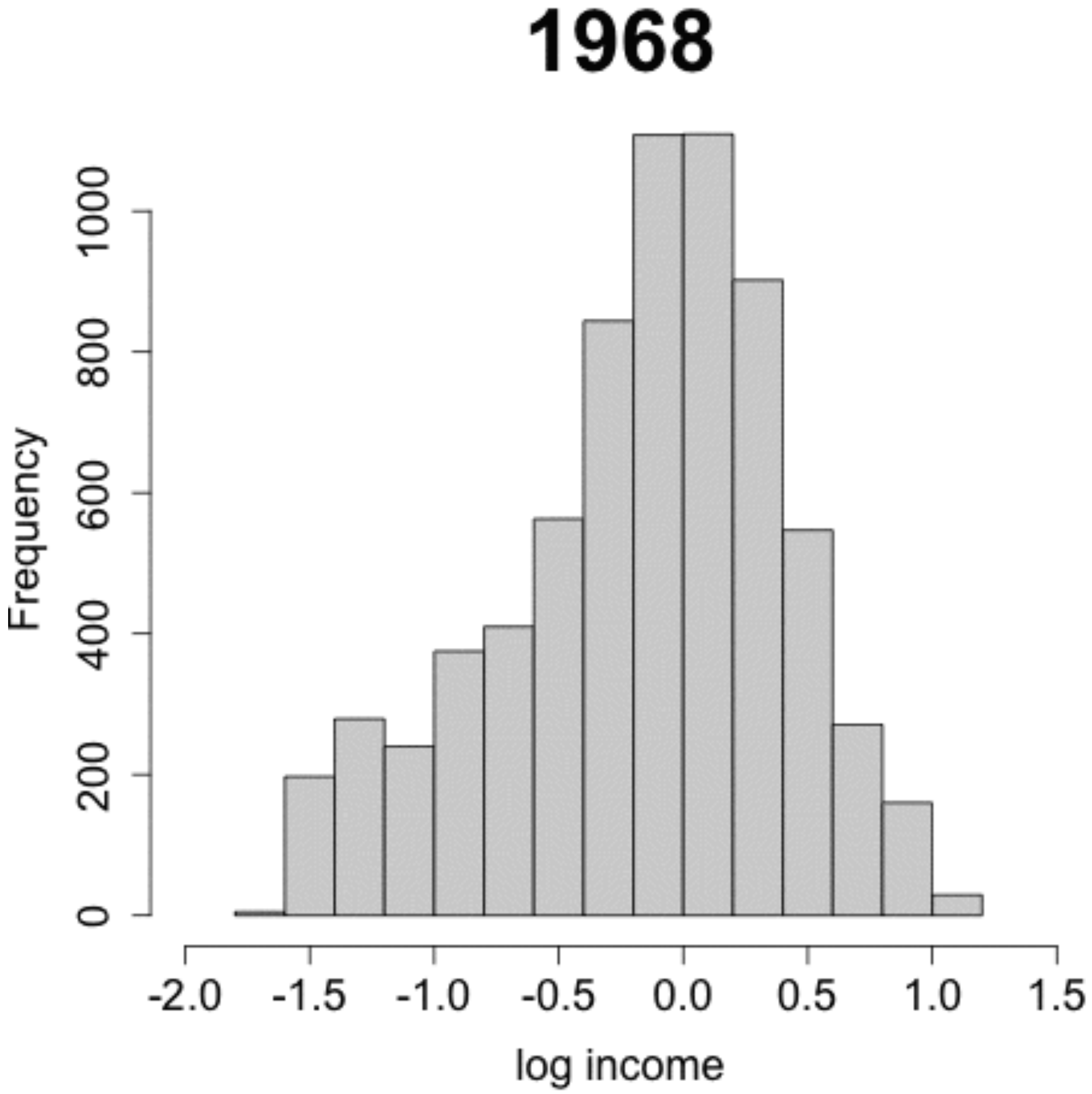
Share some common features

- E.g., to study the evolution of the economic status of a country, survey data sets of household income data are collected over multiple years: G_k is the population distribution for each year.

How to analyze data like these?

Income data from UK Family Expenditure Survey



Histograms of log household relative income from 1968 to 1988.

Data source: https://archiveshub.jisc.ac.uk/search/archives/412e6ebd-8de7-3e6e-b060-34d35cffaf15.

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Parametric approaches
Choose a suitable parametric model (e.g., normal) for each of the multiple populations
Pros: good statistical efficiency
Cons: consequence of model misspecification may be serious
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Parametric approaches	Nonparametric approaches
Choose a suitable parametric model (e.g., normal) for each of the multiple populations	Do not place distributional assumptions on the populations
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Cons: consequence of model misspecification may be serious	Cons: low statistical efficiency
No ☹️	No 🙁

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Parametric approaches	Nonparametric approaches	A Semiparametric approach
Choose a suitable parametric model (e.g., normal) for each of the multiple populations	Do not place distributional assumptions on the populations	Do not place parametric assumptions on each population
Pros: good statistical efficiency	Pros: free from the risk of model misspecification	**Model the connection between the multiple population distributions**
Cons: consequence of model misspecification may be serious	Cons: low statistical efficiency	A flexible & efficient compromise between parametric and nonparametric approaches
No ☹️	No 🙁	Yes! 😃

A Semiparametric Model: Density Ratio Model

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- Sample from G_k forms a biased sample from G_0 characterized by the exponential tilting!

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- DRM is flexible: G_0 is unspecified, allowing it to cover many distribution families.

Distribution family	**Basis function $\boldsymbol{q}(x)$**
Normal	(x, x^2)
Gamma	$(x, \log x)$
Exponential family	Sufficient statistics
…	…

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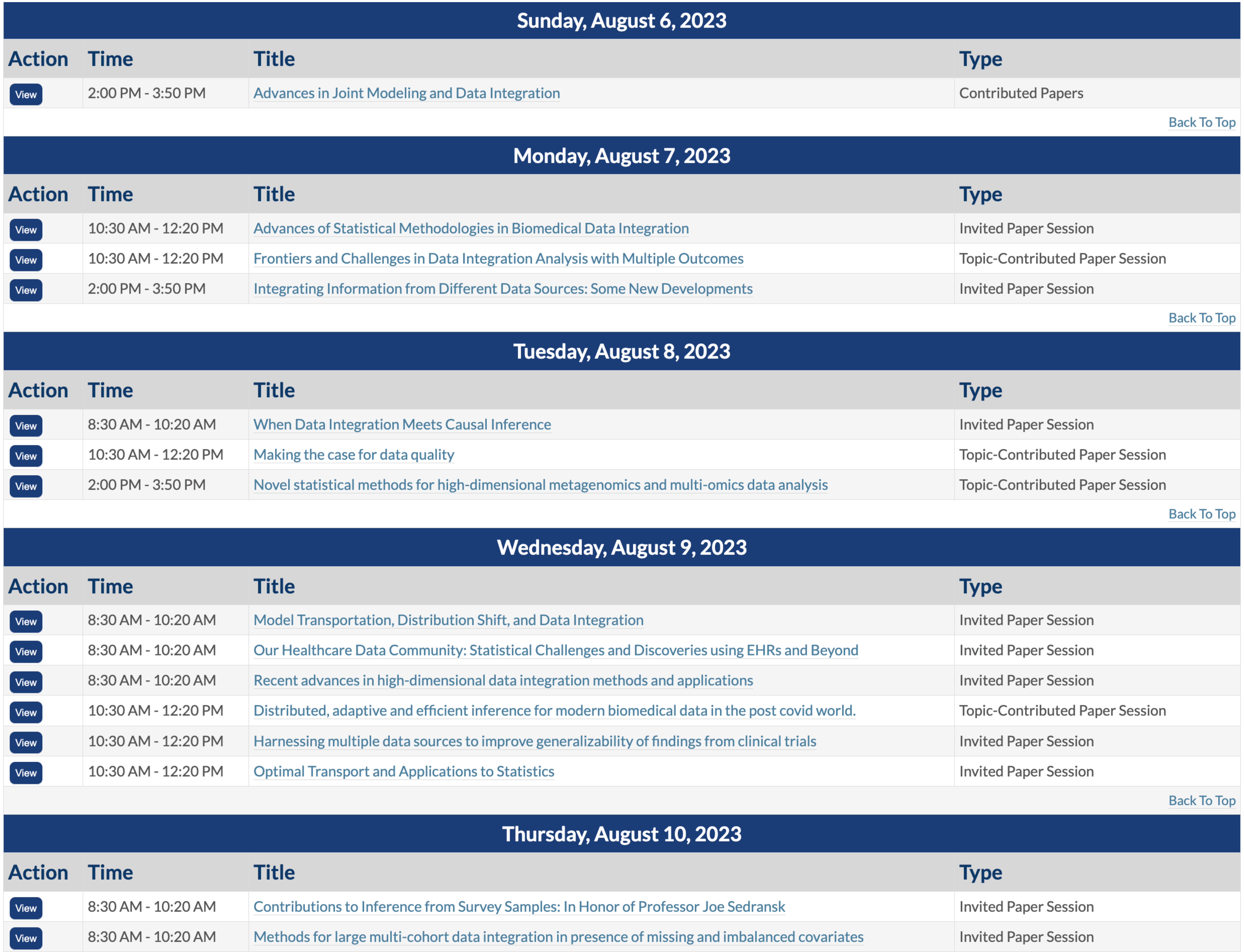
- Every G_k can be seen as a distributional shift version of G_0!
  - Can be useful for integrating data from **connected** sources/domains.

Search “data integration” in JSM2023 program…popular recently!



Sunday, August 6, 2023

Action	Time	Title	Type
View	2:00 PM - 3:50 PM	Advances in Joint Modeling and Data Integration	Contributed Papers

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Monday, August 7, 2023

Action	Time	Title	Type
View	10:30 AM - 12:20 PM	Advances of Statistical Methodologies in Biomedical Data Integration	Invited Paper Session
View	10:30 AM - 12:20 PM	Frontiers and Challenges in Data Integration Analysis with Multiple Outcomes	Topic-Contributed Paper Session
View	2:00 PM - 3:50 PM	Integrating Information from Different Data Sources: Some New Developments	Invited Paper Session

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Tuesday, August 8, 2023

Action	Time	Title	Type
View	8:30 AM - 10:20 AM	When Data Integration Meets Causal Inference	Invited Paper Session
View	10:30 AM - 12:20 PM	Making the case for data quality	Topic-Contributed Paper Session
View	2:00 PM - 3:50 PM	Novel statistical methods for high-dimensional metagenomics and multi-omics data analysis	Topic-Contributed Paper Session

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Wednesday, August 9, 2023

Action	Time	Title	Type
View	8:30 AM - 10:20 AM	Model Transportation, Distribution Shift, and Data Integration	Invited Paper Session
View	8:30 AM - 10:20 AM	Our Healthcare Data Community: Statistical Challenges and Discoveries using EHRs and Beyond	Invited Paper Session
View	8:30 AM - 10:20 AM	Recent advances in high-dimensional data integration methods and applications	Invited Paper Session
View	10:30 AM - 12:20 PM	Distributed, adaptive and efficient inference for modern biomedical data in the post covid world.	Topic-Contributed Paper Session
View	10:30 AM - 12:20 PM	Harnessing multiple data sources to improve generalizability of findings from clinical trials	Invited Paper Session
View	10:30 AM - 12:20 PM	Optimal Transport and Applications to Statistics	Invited Paper Session

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Thursday, August 10, 2023

Action	Time	Title	Type
View	8:30 AM - 10:20 AM	Contributions to Inference from Survey Samples: In Honor of Professor Joe Sedransk	Invited Paper Session
View	8:30 AM - 10:20 AM	Methods for large multi-cohort data integration in presence of missing and imbalanced covariates	Invited Paper Session

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Owen (2001): "EL keeps the effectiveness of likelihood methods and does not impose a known family distribution on the data."

Estimation Efficiency under the Density Ratio Model

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- Is it likely that the DRM-based estimators can be as efficient as the parametric estimators?

- Or When?

A two-sample scenario

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Consider two samples of sizes n_0, n_1 from G_0, G_1, with $n_0 \gg n_1$.

DRM that connects G_0, G_1:

$$g_1(x) = g_0(x)\ \exp\{\alpha + \boldsymbol{\theta}^\top \boldsymbol{q}(x)\}\,.$$

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We therefore expect the DRM estimators for G_1 to achieve the "gold-standard"parametric efficiency!

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- We consider an exponential family model for G_1:

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- It is a parametric submodel for the two-sample DRM with the same $\boldsymbol{q}(x)$:

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- MLEs under this exponential family model are the “gold-standard” parametric estimators.

Our contribution

We theoretically prove that under the two-sample scenario, the following DRM-based estimators for G_1 achieve parametric efficiency asymptotically when $n_0/n_1 \to \infty$ as $n_0, n_1 \to \infty$:

- DRM model parameters $(\alpha, \boldsymbol{\theta})$;
- Distribution function $G_1(x)$;
- Quantiles of G_1.

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- Quantiles of G_1.

Our contribution is applicable and particularly useful in applications where one wishes to make efficient inference with a small sample, aided by another large historical sample.

Quantile Estimation

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$$\boxed{\mathrm{Var}(\hat{\xi}_p)} = \frac{1}{k+1}\boxed{\frac{p(1-p)}{n_1 g_1^2(\xi_p)}} + \frac{k}{k+1}\boxed{\mathrm{Var}(\tilde{\xi}_p)}.$$

Variance of DRM quantile

Variance of Nonpara. quantile

Variance of parametric quantile

Simulation with data from normal distributions

Parameter of interest: ξ_p — the pth quantile for G_1

- Generate two samples both from $N(0,1)$.
- Obtain the DRM quantile estimator only assuming the knowledge of the most appropriate $\boldsymbol{q}(x) = (x, x^2)^\top$.
- Two competitors that only use sample from G_1:
    - MLE of quantile derived under the normal model
    - Nonparametric empirical quantile

Performance of quantile estimators

Biases are $\times\sqrt{n_1}$; Variances are $\times n_1$; Based on 1000 repetitions

Levels p	DRM-based		MLE		Nonparametric	
	Bias	Var	Bias	Var	Bias	Var
	$n_0 = k \times n_1, \quad n_1 = 1000, \quad k = 10$					
0.01	−0.02	4.91	0.03	3.81	0.51	13.53
0.05	−0.01	2.61	0.02	2.42	0.09	4.53
0.10	0.00	1.98	0.02	1.87	0.03	3.29
0.50	0.00	1.10	0.01	1.03	0.05	1.58
	$n_0 = k \times n_1, \quad n_1 = 1000, \quad k = 100$					
0.01	−0.06	3.94	−0.05	3.83	0.58	13.66
0.05	−0.05	2.46	−0.05	2.41	0.11	4.45
0.10	−0.05	1.86	−0.05	1.85	0.01	2.88
0.50	−0.05	0.97	−0.04	0.96	−0.04	1.52

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1. As $\boxed{k}$ ↑, variances of the DRM estimators approach those of the MLEs.

Matches our theoretical result!

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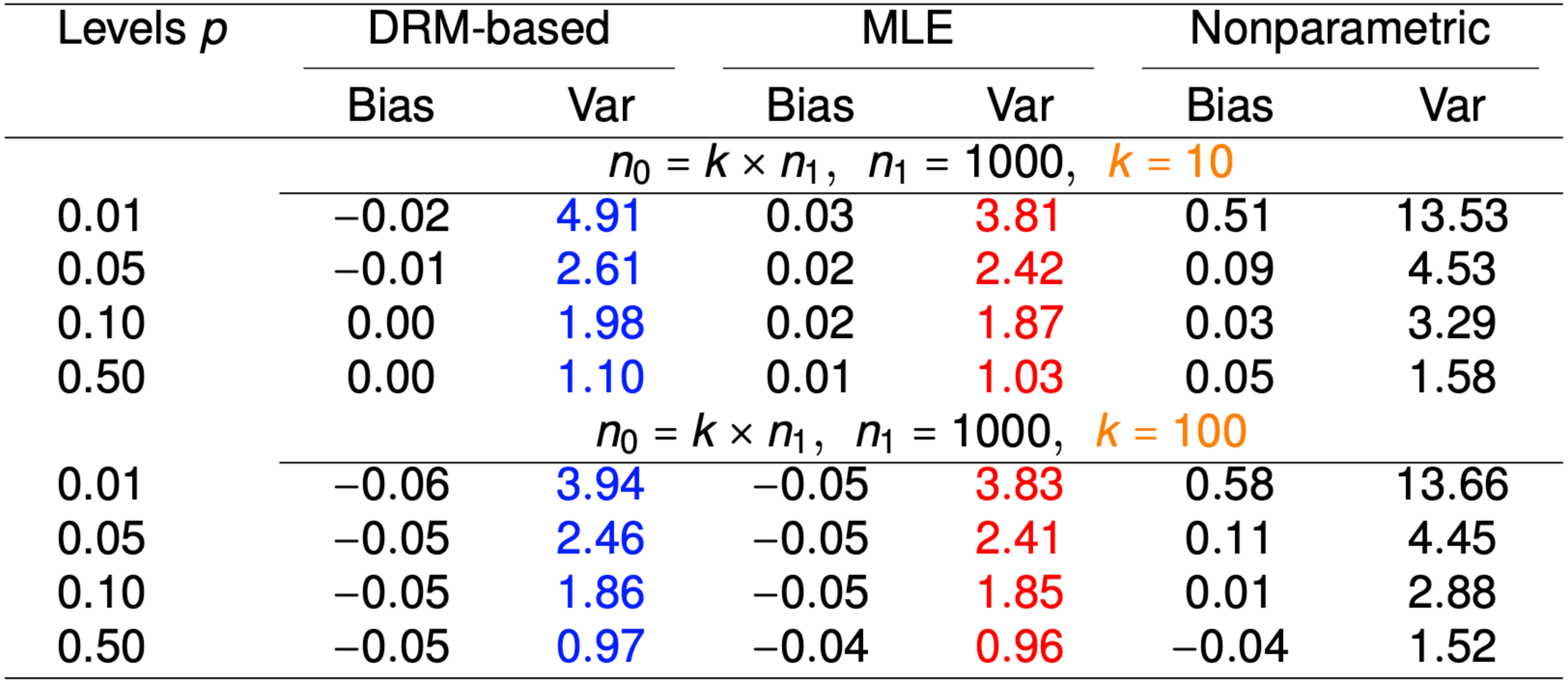

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2. Our “weighted average” result is also well supported.

Summary

- We prove that in the two-sample scenario where $n_0/n_1 \to \infty$, some DRM estimators for G_1 achieve parametric efficiency.
- Our contribution is new and particularly useful in applications where we have one large historical sample and one small sample to make inference on.
- Simulation results on quantile estimation support our theoretical findings.

References

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Thank you! :-)

Q & A